1: Direct Proofs

a) *The Philadelphia Proofs are a little-known baseball team that recently played against the Chicago Contrapositives. In their game, there were nine innings, and the Proofs outscored the Contrapositives in at least 3 but not all of the innings. → (0) For simplicity, we will refer to outscoring in the inning as "winning" the inning. The following is true of the innings:

i) They did not win both the 2nd and 5th inning.
ii) If they won the 7th or 4th inning, then they won every inning.
iii) If they won the 2nd inning, then they won the 5th inning.
iv) If they won the 3rd inning, then they won the 7th inning.
v) If they won the 1st inning or the 6th inning, then they did not win the 9th inning.
vi) If they won the 1st inning or the 5th inning, then they did not win the 8th inning.

Prove that they won exactly 3 of the innings by finding the three innings they won.

Proof. Assume (0) and (i)–(vi). Show they won 3 of the innings.

Take contrapositive of (ii) ⇒ If they didn't win every inning, they did not win the 7th or the 4th inning. Since they didn't win every inning, they didn't win either the 7th or 4th inning.

By (iii), if they win the 2nd, they win 2nd and 5th. By (i), they didn't win 2nd and 5th. Thus, they couldn't have won the second.

Take contrapositive of (iv) $\Rightarrow$ If they didn't win 7th, they didn't win 3rd. They didn't win 7th, so they didn't win 4th.

Take contrapositive of (v) $\Rightarrow$ if they win 9th inning, they don't win 1st or 6th inning. Since they won at least three innings, this leaves 5th and 8th inning. But if they win 5th, they didn't win 8th. This means 9th can't be one of the innings. Symmetric logic shows the 8th can't be one of the innings either. This leaves 1st, 5th, and 6th as possible innings. $\boxtimes$

2: Contrapositives

a) *Show that for all integers x, if $x^2 - 6x + 5$ is even, then x is odd.

Proof. Prove the contrapositive: if x is not odd then x^2-6x+5 is not even $\Leftrightarrow$ if x is even, then x^2-6x+5 is odd.

Assume x is even. Show x^2-6x+5 is odd.

$x = 2m$ for some int m

$$\begin{aligned} x^2-6x+5 &= (2m)^2-6(2m)+5 \\ &= 4m^2-12m+4+1 \\ &= 2(2m^2-6m+2)+1 \Rightarrow x^2-6x+5 \text{ is odd } \boxtimes \end{aligned}$$

b) Challenge: Show that for all real numbers x and y, if $y^3 + yx^2 \leq x^3 + xy^2$, then y ≤ x.

Proof. Prove the contrapositive: if $y > x$, then $y^3 + yx^2 > x^3 + xy^2$.

Assume $y > x$. Show $y^3 + yx^2 > x^3 + xy^2$.

$$y > x$$

$$y(x^2+y^2) > x(x^2+y^2$$

$$y^3 + yx^2 > x^3 + xy^2 \quad \boxtimes$$

c) Show that for all real numbers r, if r is irrational, then $r^{1/5}$ is irrational.

Proof. Prove the contrapositive: if $r^{1/5}$ is rational, r is rational.

Assume $r^{1/5}$ is rational. Show r is rational.

$r^{1/5} = \frac{m}{n}$ for ints m and n

$(r^{1/5})^5 = \left(\frac{m}{n}\right)^5$

$r = \frac{m^5}{n^5}$ Since m and n are int, m^5 and n^5 are ints, so r is rational. $\boxtimes$

3: If and Only Ifs

a) *Let ⌊x⌋ be the greatest integer less than or equal to x. Let ⌈x⌉ be the least integer greater than or equal to x. Show that ⌊x⌋ = ⌈x⌉ iff x is an integer.

Proof. Show if $\lfloor x \rfloor = \lceil x \rceil$, x is int and if x is int, $\lfloor x \rfloor = \lceil x \rceil$.

If $\lfloor x \rfloor = \lceil x \rceil$, x is an int:

Assume $\lfloor x \rfloor = \lceil x \rceil$. Show x is an int.

By def of $\lfloor x \rfloor$, $\lfloor x \rfloor \leq x$.

By def of $\lceil x \rceil$, $\lceil x \rceil \geq x$

$\lfloor x \rfloor \leq x \leq \lceil x \rceil$ and $\lceil x \rceil = \lfloor x \rfloor$

Since x is between two numbers that are equal, x is equal to both. $\lceil x \rceil = x$, and $\lceil x \rceil$ is an int by definition, so x must be an int. ✓

If x is an int, $\lfloor x \rfloor = \lceil x \rceil$

By def, if x is int, $\lfloor x \rfloor = x = \lceil x \rceil$ ✓

$\boxtimes$

$x^2+ax+b=0$

b) Show that ~~$x^2 + ax + b$~~ has two distinct real solutions iff $a^2 - 4b > 0$.

Proof. Show if $x^2+ax+b=0$ has 2 real sol'n, $a^2-4b>0$.
Show if $a^2-4b>0$, $x^2+ax+b=0$ has 2 real sol'n.

Subproof: if $x^2+ax+b=0$ has 2 real sol'n, $a^2-4b>0$.
Assume $x^2+ax+b=0$ has 2 real sol'n. Show $a^2-4b>0$

By quadratic formula, $x_1 = \dfrac{-a+\sqrt{a^2-4b}}{2}$, $x_2 = \dfrac{-a-\sqrt{a^2-4b}}{2}$

Since sol'ns are real, $a^2-4b \geq 0$, since if it wasn't $\sqrt{a^2-4b}$ would be imaginary.

Sol'ns are distinct, so $\dfrac{-a+\sqrt{a^2-4b}}{2} \neq \dfrac{-a-\sqrt{a^2-4b}}{2}$.

$\sqrt{a^2-4b} \neq -\sqrt{a^2-4b}$

$2\sqrt{a^2-4b} \neq 0$

$\sqrt{a^2-4b} \neq 0$

$a^2-4b \neq 0$, so $a^2-4b>0$. ✓

Subproof. if $a^2-4b>0$, $x^2+ax+b=0$ has 2 real sol'n.

Assume $a^2-4b>0$. Show x^2+ax+b has 2 real sol'n.

By quadratic formula, $x_1 = \dfrac{-a \pm \sqrt{a^2-4b}}{2}$, $x_2 = \dfrac{-a-\sqrt{a^2-4b}}{2}$

Since $a^2-4b>0$, $\sqrt{a^2-4b}$ is real so x_1 and x_2 are real.

Show if $a^2-4b>0$, x_1 and x_2 are distinct. Use proof by contraposition: if x_1 and x_2 are not distinct, $a^2-4b \le 0$.

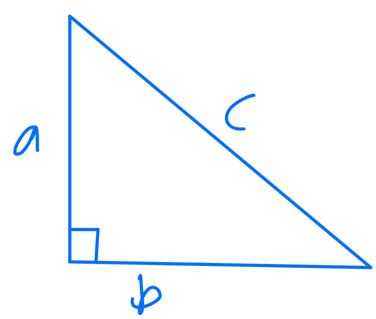
Assume $x_1 = x_2$. Show $a^2-4b \le 0$.

$$\frac{-a+\sqrt{a^2-4b}}{2} \neq \frac{-a-\sqrt{a^2-4b}}{2} \Rightarrow a^2-4b=0 \rightarrow a^2-4b \le 0$$

So x_1 and x_2 are real and distinct. ✓

4: Proof by Contradiction

a) *Use the Pythagorean theorem to prove that the length of the hypotenuse of a right triangle must be less than the sum of the length of the legs.



Pythagorean theorem: $a^2+b^2=c^2$

Proof. Assume Pythagorean theorem. Show $c<a+b$

Assume $c \ge a+b$ for contradiction.

$(c)^2 \ge (a+b)^2$

$c^2 \ge a^2+2ab+b^2$

$c^2 \ge (a^2+b^2)+2ab$

$c^2 \ge c^2+2ab$ (Pythagorean theorem)

$0 \ge 2ab$

a and b are side lengths, so $a > 0$ and $b > 0$. Thus, $ab > 0$, and $2ab > 0$. But we just showed that $2ab \leq 0$. This is a contradiction.

$c < a+b$. $\boxtimes$

b) Prove that an integer cannot be both even and odd.

Assume that there is an integer that is both even and odd. Refer to this integer as i.

That means there exists integers m, n such that $i = 2m$ and $i = 2n+1$.

$$2m = 2n+1$$
$$m = n + \frac{1}{2}$$
$$m - n = \frac{1}{2}$$

If you take the difference of two integers, it must be an integer. However, $m-n = \frac{1}{2}$, which is not an integer. This creates a contradiction.

No integer is both even and odd. $\boxtimes$

c) Prove that there is no least positive rational number.

Proof. Assume there is a least positive rational number for contradiction. Refer to this number as r. There exists integers m, n s.t. $r = \frac{m}{n}$.

Let $a = \frac{r}{2} = \frac{m}{2n}$

Since r is positive, a is less than r, and can be written as a ratio of integers, meaning a is a positive rational number less than r. But r is the least positive rational number. This creates a contradiction. There is no least positive rational number. $\boxtimes$

d) Challenge: Prove that $\log_2(3)$ is irrational.

Proof. Assume $\log_2(3)$ is rational for contradiction.

There exists integers m, n s.t. $\log_2(3) = \frac{m}{n}$

$2^{m/n} = 3$ (def of log)

$\left(2^{m/n}\right)^n = (3)^n$

$2^m = 3^n$

If m and n are integers, 2^m is even and 3^n is odd, since even $\times$ even $=$ even and odd $\times$ odd $=$ odd. This means the same number is both even and odd, which is a contradiction. $\log_2(3)$ is irrational. $\boxtimes$